

MATHEMATICS

NOTE ON LILL'S METHOD OF SOLUTION OF NUMERICAL EQUATIONS

BY

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§ 1.

In 1867 LILL [1] gave a graphical method to resolve the real roots of a numerical equation. The main principle in this method is the representation of a polynomial with real coefficients by a right-angled set of sides, called *orthogones*, where the lengths of the consecutive sides are proportional to the values of the coefficients of the polynomial, and where the directions in which these sides are drawn are dependent on the variations of sign of these coefficients. The value of the polynomial $f(z)$ for each real value of z can be determined with reasonable accuracy.

In the present note I show that this method of representating of a polynomial by an orthogone can be extended. The condition that the consecutive sides are at right angles can be released. Furthermore it is possible to represent in the same figure the polynomials in the variable z : $\frac{f(z) - f(x)}{z - x}$ and $f(x + z)$ for a fixed value of x , and to determine graphically the values of $f'(x)$, $f''(x)$, \dots , $f^{(n)}(x)$.

§ 2.

We consider a polynomial of degree n

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n \quad (a_0 \neq 0),$$

where a_i ($i = 0, 1, \dots, n$) are real numbers.

Now we construct a polygonal set of sides with lengths proportional to the a_i respectively. Each pair of consecutive sides include a fixed angle φ . A rule for the construction of the set of sides may be stated as follows. If two successive coefficients a_i and a_{i+1} have different signs (a variation) the direction of the side represented by a_{i+1} (the a_{i+1} -side) is obtained by a rotation of the a_i -side over an angle φ in positive sense (that is counter-clockwise). If a_i and a_{i+1} have same signs, the direction of the a_{i+1} -side is obtained by a rotation of the a_i -side over an angle $180^\circ - \varphi$ in negative sense. In the case of a zero coefficient a_i the direction of the following a_{i+1} -side can be obtained by giving a_i a very small value $+$ or $- \varepsilon$, which tends to zero.

Hence the segment $\overline{A_n P_{n+1}}$ represents the value of $f(x)$ with (1). Is $\overline{A_n P_{n+1}} = 0$, then x is a root of the equation $f(z) = 0$. When it is possible to determine A_1 such that the endpoints of the two polygonparts coincide, then a root of $f(z)$ is found. In the case $\varphi = \pi/2$ LILL constructed a simple instrument to find the resolving polygonparts in order to determine the real roots of a given numerical equation.

§ 3.

If we choose on $P_1 P_2$ another point $B_1 \neq A_1$, with $y = -\frac{\overline{P_1 B_1}}{a_0}$ and we construct the corresponding polygonpart B_1, B_2, \dots, B_n , we find a segment $\overline{B_n P_{n+1}}$ representing $f(y)$, so that:

$$\begin{aligned} \overline{B_n P_{n+1}} &= f(y) = a_0 y^n + a_1 y^{n-1} + \dots + a_{n-1} y + a_0 \\ &= a_0 y^n + (\overline{A_1 P_2} - \overline{A_1 P_1}) y^{n-1} + (\overline{A_2 P_3} - \overline{A_2 P_2}) y^{n-2} + \dots + (\overline{A_n P_{n+1}} - \overline{A_n P_n}) \\ &= a_0 y^n + (\overline{A_1 P_2} - a_0 x) y^{n-1} + (\overline{A_2 P_3} - \overline{A_1 P_2} x) y^{n-2} + \dots + \overline{A_n P_{n+1}} - \overline{A_{n-1} P_n} x \\ &= (y-x) \{a_0 y^{n-1} + \overline{A_1 P_2} y^{n-2} + \overline{A_2 P_3} y^{n-3} + \dots + \overline{A_{n-1} P_n}\} + \overline{A_n P_{n+1}}. \end{aligned}$$

Since $\overline{A_n P_{n+1}} = f(x)$, we have

$$\overline{B_n A_n} = f(y) - f(x) = (y-x) \{a_0 y^{n-1} + \overline{A_1 P_2} y^{n-2} + \overline{A_2 P_3} y^{n-3} + \dots + \overline{A_{n-1} P_n}\}.$$

From the conformity of the triangles mentioned above it follows:

$$\frac{a_0}{\overline{P_0 A_1}} = \frac{\overline{A_1 P_2}}{\overline{A_1 A_2}} = \frac{\overline{A_2 P_3}}{\overline{A_2 A_3}} = \dots = \frac{\overline{A_{n-1} P_n}}{\overline{A_{n-1} A_n}}.$$

Hence

$$f(y) - f(x) = (y-x) (\overline{P_0 A_1} y^{n-1} + \overline{A_1 A_2} y^{n-2} + \dots + \overline{A_{n-1} A_n}) \frac{a_0}{\overline{P_0 A_1}},$$

and so:

$$\overline{P_0 A_1} y^{n-1} + \overline{A_1 A_2} y^{n-2} + \dots + \overline{A_{n-1} A_n} = \frac{\overline{P_0 A_1}}{a_0} \frac{f(y) - f(x)}{y - x}.$$

This implies that the polygonpart $P_0 A_1 \dots A_n$ represents the polynomial $\frac{f(y) - f(x)}{y - x}$, multiplied by a factor $\frac{\overline{P_0 A_1}}{a_0}$.

If a real root x of $f(z) = 0$ is found by a resolving polygonpart, then this polygonpart, as representing the polynomial of lower degree $\frac{f(z)}{z - x}$ can be used to find a next root of $f(z) = 0$.

The representation of the polynomials $f(z)$ and $\frac{f(z) - f(x)}{z - x}$ is very useful to demonstrate several properties of the real roots and of the relations between the roots and the coefficients, also to deduce the discriminants of quadratic, cubic and quartic equations, a.s.o.

§ 4.

Let P_0, P_1, \dots, P_{n+1} (fig. 2 with $n = 4$) be a polygonpart with angle φ , representing the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n.$$

If $x = -\frac{\overline{P_1 A_1}}{a_0}$, then $f(x) = \overline{A_n P_{n+1}}$, and P_0, A_1, \dots, A_n represents the polynomial

$$\Theta(z) = \frac{\overline{P_0 A_1}}{a_0} \frac{f(z) - f(x)}{z - x}.$$

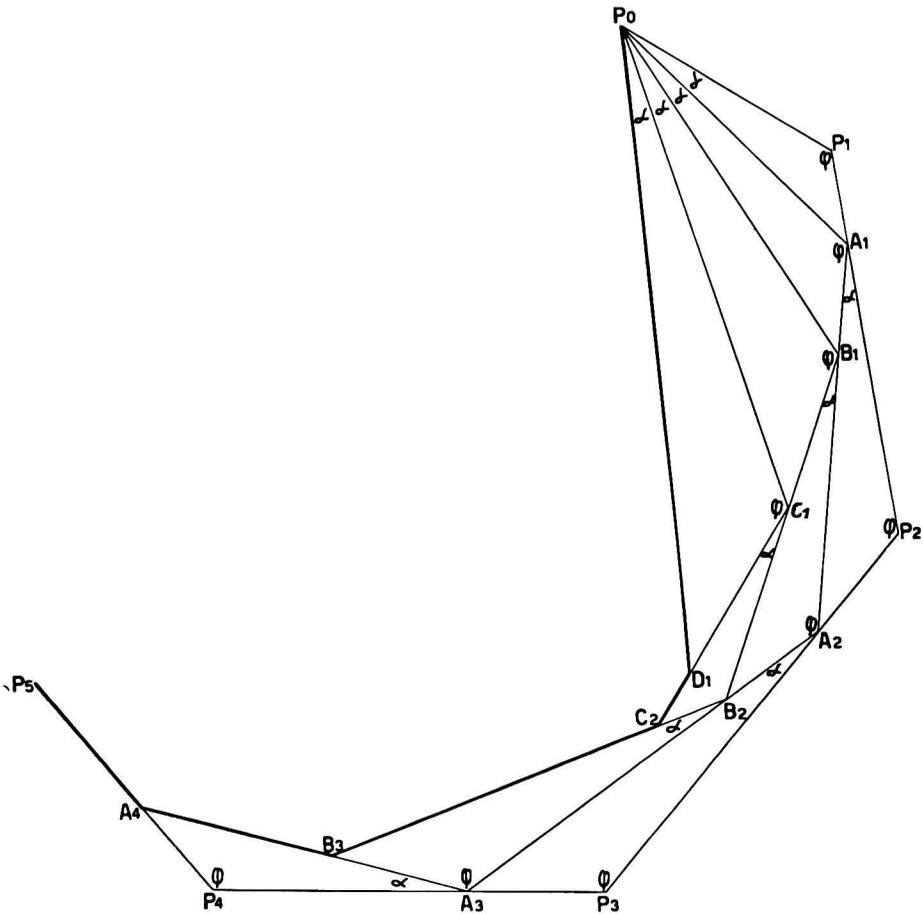


Fig. 2

When we construct in this polygonpart a new polygonpart P_0, B_1, \dots, B_{n-1} with angle φ corresponding with the point B_1 on the side $A_1 A_2$, so that

$$x_1 = -\frac{\overline{A_1 B_1}}{P_0 A_1}, \text{ then } \overline{B_{n-1} A_n} = \Theta(x_1) = \frac{\overline{P_0 A_1}}{a_0} \frac{f(x_1) - f(x)}{x_1 - x},$$

and P_0, B_1, \dots, B_{n-1} represents the polynomial

$$\psi(z) = \frac{\overline{P_0 B_1}}{\overline{P_0 A_1}} \frac{\Theta(z) - \Theta(x_1)}{z - x_1} = \frac{\overline{P_0 B_1}}{a_0} \frac{f(z) - f(x)}{z - x} \frac{f(x_1) - f(x)}{x_1 - x}.$$

If we choose B_1 such that $\angle P_1 P_0 A_1 = \angle A_1 P_0 B_1 = \alpha$, then it follows from the conformity of the triangles $P_1 P_0 A_1$ and $A_1 P_0 B_1$:

$$x = -\frac{\overline{P_1 A_1}}{a_0} = -\frac{\overline{A_1 B_1}}{\overline{P_0 A_1}} = x_1, \quad \overline{P_0 B_1} = \frac{\overline{P_0 A_1}^2}{a_0},$$

$$\overline{B_{n-1} A_n} = \lim_{z \rightarrow x} \Theta(z) = \frac{\overline{P_0 A_1}}{a_0} f'(x).$$

In this case P_0, B_1, \dots, B_{n-1} represents the polynomial:

$$\psi(z) = \frac{\overline{P_0 B_1}}{a_0} \frac{f(z) - f(x)}{z - x} \frac{f'(x)}{z - x} = \left(\frac{\overline{P_0 A_1}}{a_0}\right)^2 \frac{f(z) - f(x)}{z - x} \frac{f'(x)}{z - x}.$$

Proceeding with a point C_1 on the side $B_1 B_2$ with $\angle B_1 P_0 C_1 = \alpha$, it is easily seen that

$$C_{n-2} B_{n-1} = \lim_{z \rightarrow x} \psi(z) = \left(\frac{\overline{P_0 A_1}}{a_0}\right)^2 \frac{f''(x)}{2!}.$$

Finally we find sides with lengths

$$\left(\frac{\overline{P_0 A_1}}{a_0}\right)^k \frac{f^{(k)}(x)}{k!} \quad (k = 0, 1, \dots, n).$$

Hence for a fixed value of x it is possible to find graphically the values of $f^{(k)}(x)$ ($k = 0, 1, \dots, n$). Furthermore it is evident that the angles between each pair of these consecutive sides are $\varphi + \alpha$, so that these sides form a new polygonpart with angle $\varphi + \alpha$ (in fig. 2 $P_5, A_4, B_3, C_2, D_1, P_0$), representing the polynomial:

$$\begin{aligned} F(z) &= f(x) + \left(\frac{\overline{P_0 A_1}}{a_0}\right) \frac{f'(x)}{1!} z + \left(\frac{\overline{P_0 A_1}}{a_0}\right)^2 \frac{f''(x)}{2!} z^2 + \dots + \left(\frac{\overline{P_0 A_1}}{a_0}\right)^n \frac{f^{(n)}(x)}{n!} z^n \\ &= f\left(x + \frac{\overline{P_0 A_1}}{a_0} z\right). \end{aligned}$$

From

$$\frac{\overline{P_0 A_1}}{a_0} = \frac{\sqrt{a_0^2 + \overline{P_1 A_1}^2 - 2a_0 \overline{P_1 A_1} \cos \varphi}}{a_0} = \sqrt{1 + x^2 + 2x \cos \varphi}$$

we have

$$F(z) = f\left(x + z \sqrt{1 + x^2 + 2x \cos \varphi}\right),$$

represented by the polygonpart $P_{n+1}, A_n, B_{n-1}, \dots, P_0$.

For the orthogones of LILL is

$$F(z) = f(x + z\sqrt{1+x^2}).$$

For small values of x it is possible to choose the angle φ in such way, that the factor $\frac{\overline{P_0 A_1}}{a_0}$ equals the unit. If $|x| < 2$ and $\varphi = bg \cos(-\frac{1}{2}x)$, then $\frac{\overline{P_0 A_1}}{a_0} = 1$, and the values of $\frac{f^{(k)}(x)}{k!}$ ($k = 0, 1, \dots, n$) may be determined directly. In this case $F(z) = f(x + z)$, and the polygonpart, representing this polynomial, has the angle $180^\circ - \varphi$.

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LITERATURE

1. LILL, Résolution graphique des équations numériques d'un degré quelconque à une inconnue. *Nouv. Ann. de Math.*, 2e série, t. 6. Paris (1867).
2. L. CREMONA, *Elemente des Graphischen Calculs* 61–64, Leipzig (1875).