Lill's Method and Graphical Solutions to Quadratic Equations

Raul Prisacariu

http://raulprisacariu.com/

Introduction

In this informal paper I want to present a few techniques that can be used to find the solutions to quadratic equations. These techniques are only useful for quadratics that have real coefficients (no complex coefficients).

The techniques presented in this paper require a knowledge of the Lill's method for representing polynomial equations. If you want to familiarize yourself with Lill's method, I recommend this paper (external website) and my paper "Lill's Method and the Sum of Arctangents". The techniques presented here were already discovered by other people. Nonetheless, in this paper I want to present most of the useful techniques that I know. After I introduce the techniques I will show an interesting property of complex solutions to quadratic equations.

The Golden Quadratic Equation

To make the presentation more interesting, I will use a special quadratic equation as an example. The equation is $G(x)=x^2 -x -1$. This quadratic equation has the roots $x_1=\varphi(\,golden\ ratio) \approx 1.618033$ and $x_2= -\,\Phi$ (negative reciprocal of golden ratio) ≈ -0.618033 . I use this quadratic equation because I assume that everybody likes the golden ratio. In Image 1 you can see the Lill representation of G(x). P_0P_1 represents the coefficient $a_2= 1$, P_1P_2 represents the coefficient $a_1= -1$ and the segment P_2P_3 represents the segment $a_0= -1$. The formula I use to obtain the Lill representation is: $a_k e^{i(n-k)\pi/2}$, where a_k is the coefficient corresponding to the segment, i is the imaginary number and $0 \le k \le n$. This is not the only valid way of representing a polynomial using Lill's method, but it is the one that I prefer.

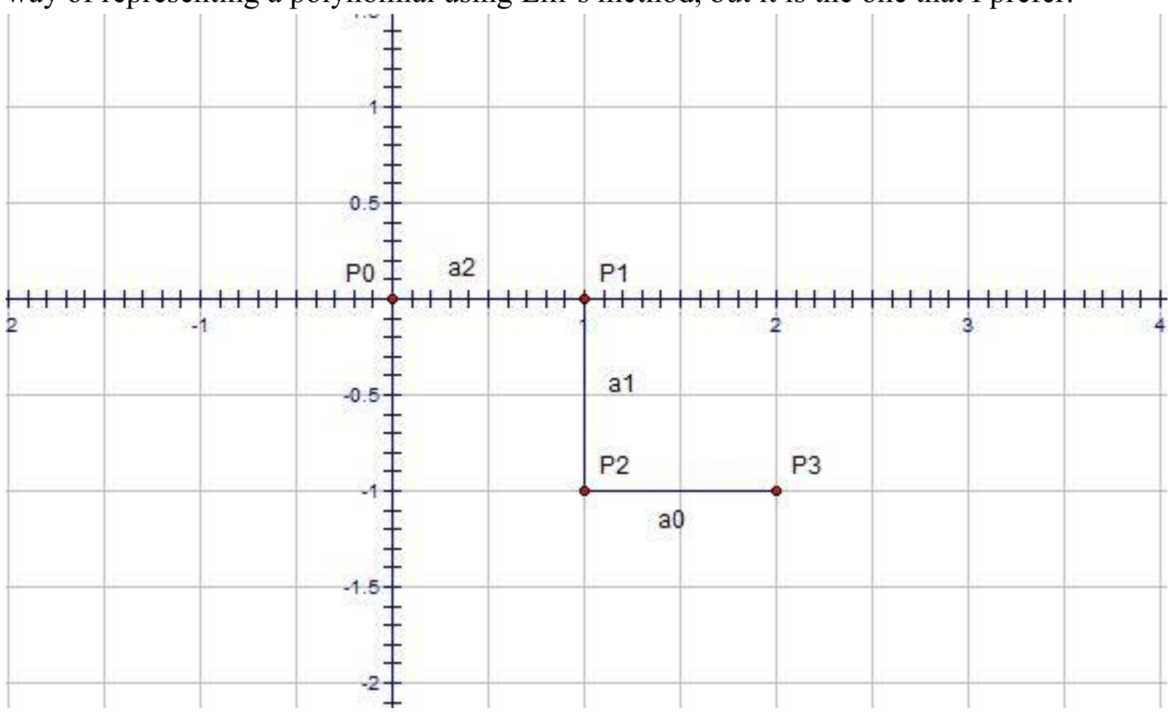

Image 1

Now that we have the basic Lill representation, I will present 2 techniques for solving P(x) in a graphical manner. In this case, I will provide 2 ways of constructing the golden ratio. This should be a bonus for the fans of the golden ratio.

Carlyle Circle/Lill Circle

The first method makes use of the Carlyle Cicle or the Lill Circle. The Carlyle Circle was discovered by Thomas Carlyle way before Eduard Lill developed his method of representing polynomial equations. The Carlyle circle was introduced by John Leslie (who was a professor of Carlyle) in his book "Elements of geometry and plane trigonometry" starting at page 176. The Carlyle Circle can easily be adapted to the Lill representation of a second degree polynomial. For convenience, I will use the term Lil circle.

For any Lill representation of a second degree polynomial P(x), the center of the Lill circle is located at the midpoint C of the segment P_0P_3 and the radius $r = P_0C = CP_3$. If P(x) has real roots, the Lill circle will intersect the extended line that passes though P_1 and P_2 at the point or points that give the solution to the quadratic equation. In Image 2 you can see the solution for G(x). You can see that the circle C intersects the line that passes through P_1 and P_2 at X_1 and X_2. You can also see that the length of segments P_1X_1 and P_1X_2 give the absolute values of the roots x_1 and x_2.

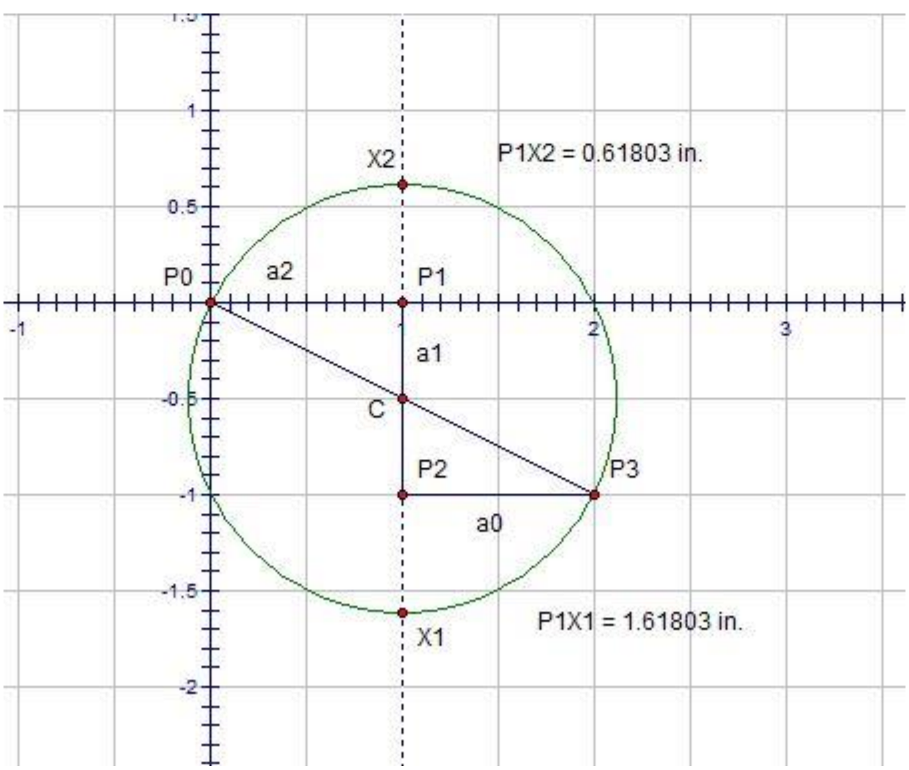

Image 2

If you read my paper for which I provided a link at the beginning of this paper, you should know how to determine the sign of a root. You should also know that a_2 acts as a scaling factor. In the case of G(x), $a_2 = 1$, so no scaling was necessary. But if we suppose that a_2 is different than 1 (but not 0), then the values of the roots would have been equal to P_1X_1/a_2 and P_1X_2/a_2.

Pappus method for solving quadratics

This is another method presented in John Leslie's book on geometry. This method was presented starting with the page 340 and it was a note to the Carlyle circle method. This construction was supposedly developed by the ancient mathematician Pappus. This method requires you to find the midpoint A of P_1P_2. Then you make a circle with the center at A and the radius equal to $AP_1 = AP_2$. Then construct the segment P_0P_3. If the polynomial P(x) has real roots, the circle A should intersect P_0P_3 at 1 or 2 points, say B and C. From B and C, you must construct the perpendicular lines to the segment P_0P_3. The solutions should be at the intersection of the perpendiculars to P_0P_3 and the extended line that passes though P_1 and P_2.

In Image 3, you can see that the perpendicular line to P_0P_3 at C, meets the extended line that passes though P_1 and P_2 at X_1. Similarly, the perpendicular at C, intersects the extended line at X_2. This should be an interesting method for constructing the golden ratio.

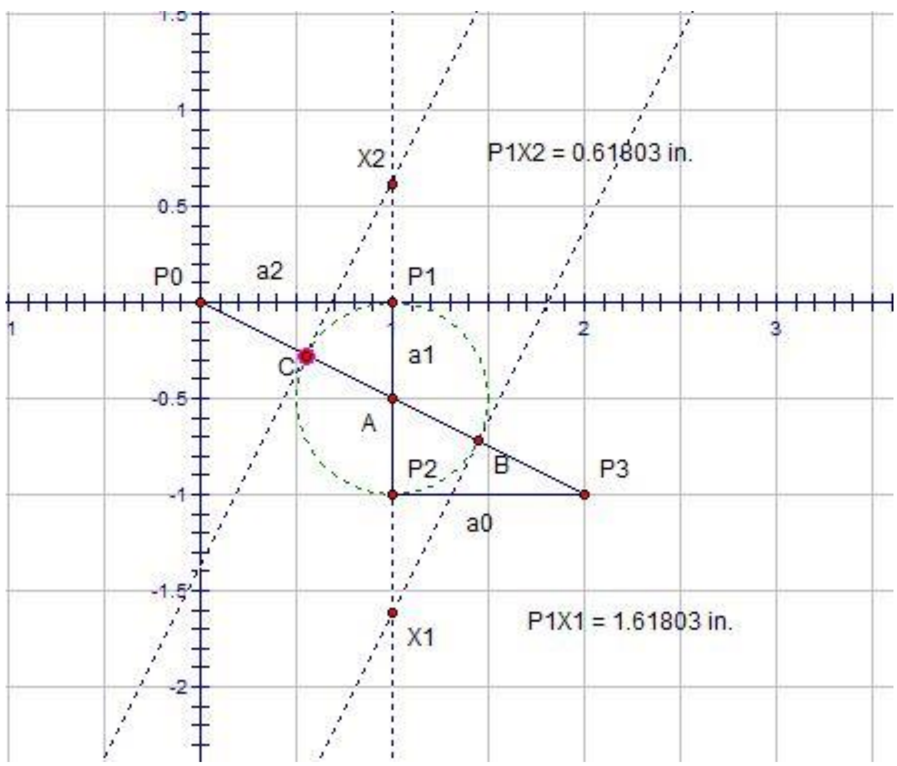

Image 3

Quadratics with complex roots

Now we want to see how to solve quadratics that have complex roots. As an example, I will use $C(x) = x^2 - 2x + 5$, with $x_1 = 1 + 2i$ and $x_2 = 1 - 2i$. What happens if we draw a Lill circle? In Image 4 we can see that the Lill circle doesn't intersect the extended line passing though the points P_1 and P_2. We can also try the Pappus method, but again will fail to get the required intersections. So, we need a new method.

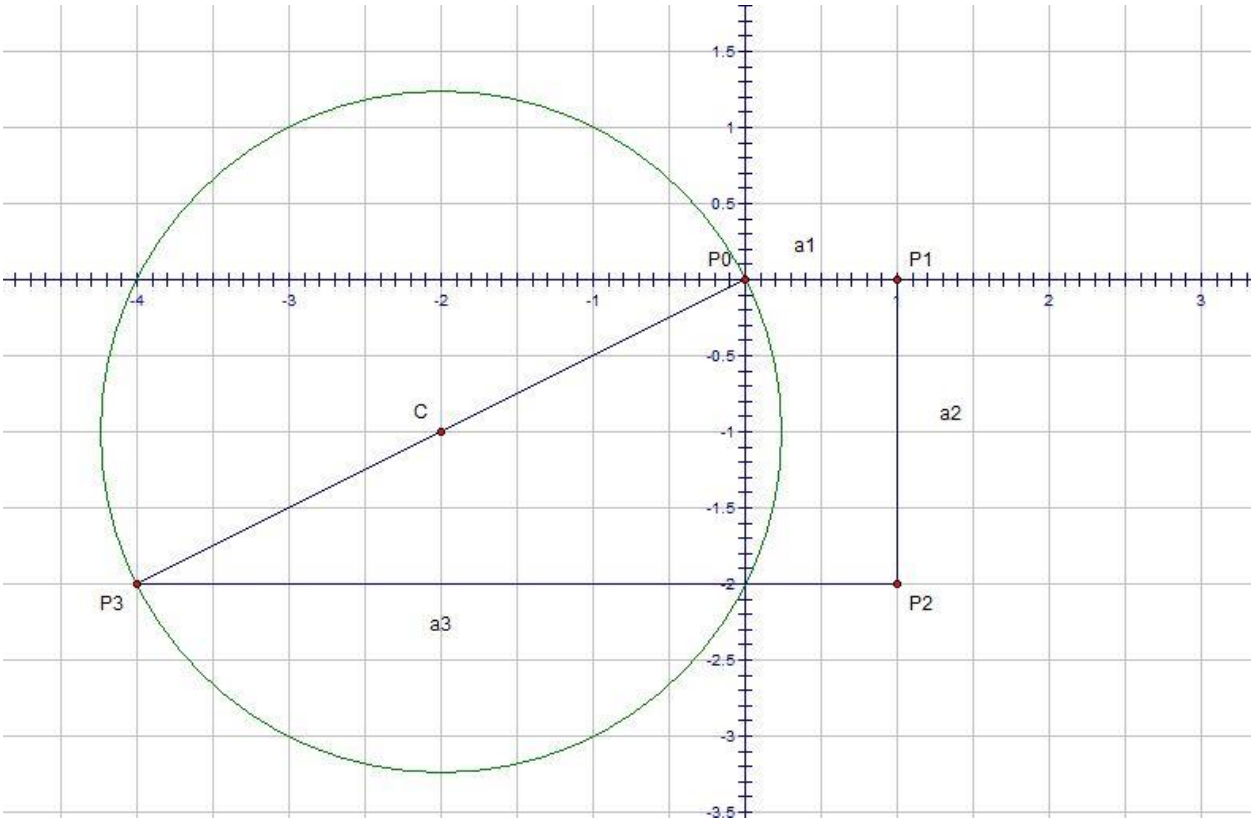

Image 4

The method for finding complex roots is a bit more complex. The first step is to construct the midpoint A of the segment P_1P_2. Then we shall construct a line that is perpendicular to P_1P_2 at the point A. Next, we will construct the segment $P_2B = abs(a_1)$, with P_2B having the opposite direction of P_2P_3. Now, we construct the midpoint M of the segment BP_3. Next, we construct the circle with the center at M and radius $r = MB = MP_3$. The circle M should intersect the extended line that passes through P_1 and P_2 at 2 points. Pick one of the points and call it D. Now construct the circle with the center at P_2 and the radius $r = P_2D$. The circle with center P_2 and radius P_2D should intersect the perpendicular line that passes though A at the points X_1 and X_2 that give the solutions to the polynomial equation with complex roots. AP_1/a_1 gives the real value of the roots, while X_1A/a_1 and X_2A/a_1 give the complex values of the roots.

Image 5 shows the graphical solution for C(x). In this case, $a_1 = 1$, so we don't need to scale the solutions. AP_1 has the length 1, and it gives the real part of x_1 and x_2. The lengths of X_1A and X_2A are both equal to 2. So, the points X_1 and X_2 are the geometric interpretations of x_1 and x_2.

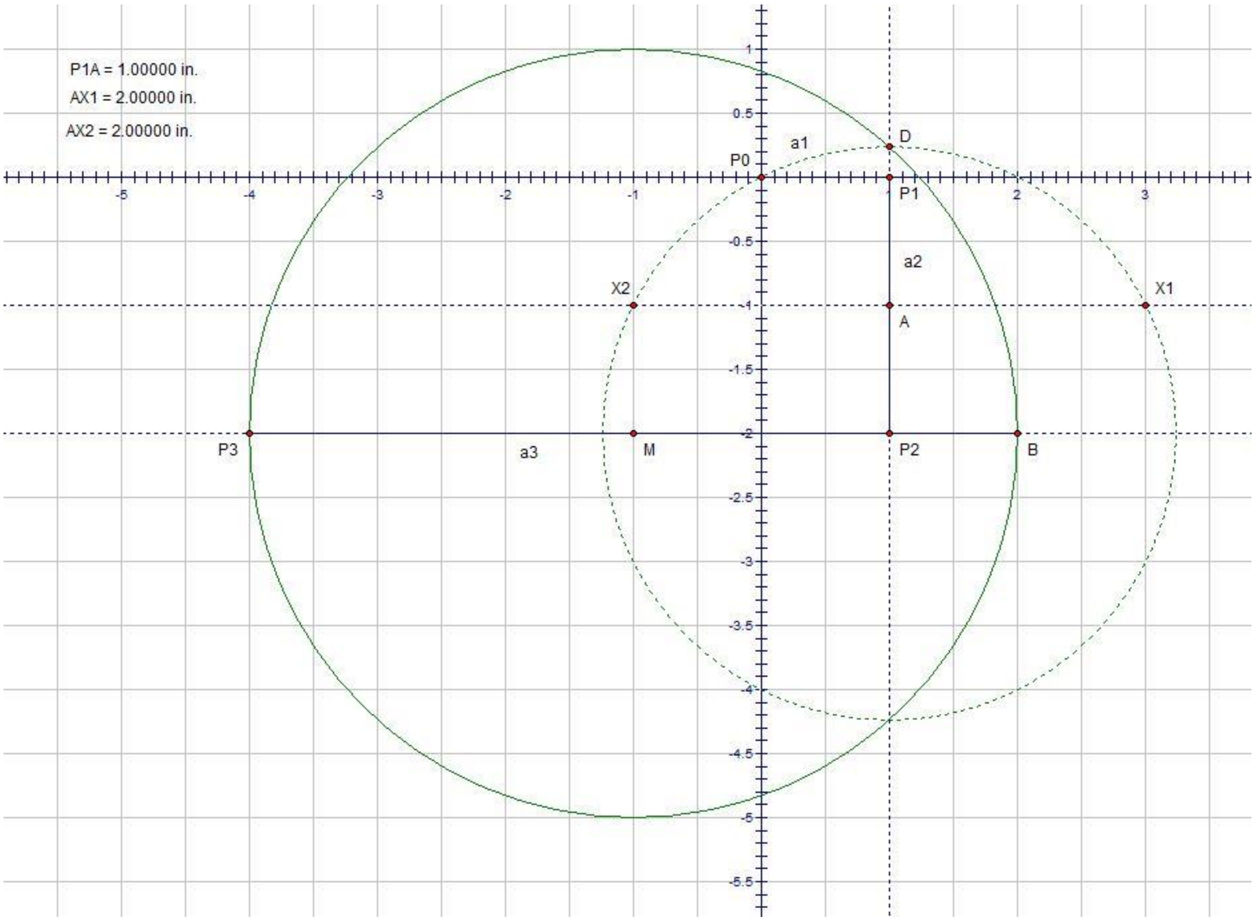

Image 5

Complex Roots and the Lill Circle Inversion Property

Even though the Lill circle was not useful for finding the complex roots, the Lill circle seems to be connected to the solution points. The property can be defined in the following way: Let P(x) be a 2^{nd} degree polynomial with real coefficients and with 2 complex roots. The Lill solutions points X_1 and X_2 that represent the roots x_1 and x_2, are in an inverse relationship with respect to the Lill circle. The only exception occurs when $x_1 = i$ and $x_2 = -i$.

In our case the center of the Lill circle is C and the radius is $r = P_0C = CP_3$. If X_1 is the inverse of X_2 with respect to the Lill circle, then $CX_1 * CX_2 = r^2$ and the points C, X_1 and X_2 are collinear (on the same line). In Image 6 I included additional calculations that show that indeed X_1 is the inverse of X_2.

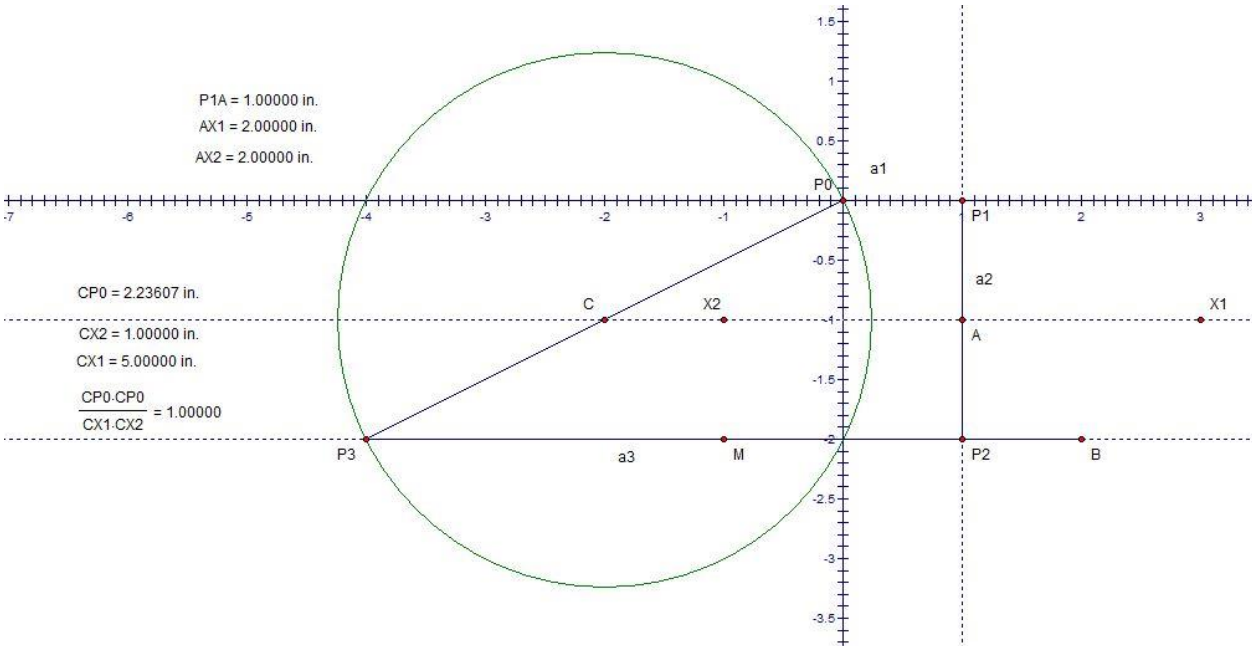

Image 6

I want to add that to my knowledge, this property is not mentioned anywhere else. Maybe in a future paper I will try to show a rigorous proof, which is beyond the scope of this paper.

Final Notes

While doing more research for this paper, I found a method that uses the Lill circle to obtain the complex solutions. The method is found in the paper "83.55 The Complex Roots of a Quadratic from a Circle" by Ladislav Beran. I also recommend the paper "Carlyle Circles and the Lemoine Simplicity of Polygon Constructions" by Duane W. DeTemple. This link discusses 2 methods explained in this paper. For more resources to study Lill's method, you can look on my website at the papers section or this Lill resources page.

I wanted to keep this informal paper short, so I avoided proofs or extended explanations. Nonetheless, I hope that I described the methods in a manner that is easy to understand. I also tried to provide many additional resources for people interested in Lill's method. One of my intentions is to show the beauty and usefulness of Lill's method.