Representing Polynomials with Complex Coefficients using Lill's Method

Abstract

In this paper we show how to represent polynomial equations with complex coefficients using Lill's method. We also discuss a few general properties.

Introduction

Lill’s method is a remarkable visual method that can be used to represent and solve polynomial equations. Lill’s method was developed in the 19th century by the Austrian engineer Eduard Lill. All the papers that discussed Lill’s method since the time of Lill himself, only dealt with polynomial equations with real coefficients. Eduard Lill showed in his second paper [1] how to represent the roots that have imaginary parts, but to our knowledge there is no paper that shows how to represent polynomials with complex coefficients using Lill’s method. The primary goal of this paper is to show that Lill’s method can be used to represent polynomials with complex coefficients.

After we show how to deal with polynomials with complex coefficients, we discuss in some detail three important general properties. Two of the general properties were discussed in other papers. However, our extension of Lill’s method gives us a better understanding of one of the properties. The third property discussed in this paper is to our knowledge not treated in other papers dealing with Lill’s method.

At the end of this paper we discuss a few reasons why Lill’s method deserves to be known by a larger audience. We also mention a few reasons why Lill’s method may prove to be a useful educational tool.

Graphing Polynomials with Real coefficients

Before we talk about polynomials with complex coefficients, it would be more convenient to discuss the polynomials with real coefficients. There are a few ways of representing a polynomial using Lill’s method, but we use a method very similar to the one described in paper [2].

So let us have the general polynomial of order n, $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, where the coefficients $\{a_n, a_{n-1}, \ldots, a_1, a_0\}$ are real numbers. In Lill's method the polynomial P(z) can be represented by a path of connected vectors $P_0\ P_1\ P_{2\ldots}\ P_n\ P_{n+1}$, where $\overrightarrow{P_kP_{k+1}}$ is a vector corresponding to a_{n-k}, where $0\leq k \leq n$. For convenience, we let P_0 to be located at (0,0). Since P_0 has a fixed location, we always start by graphing the vector $\overrightarrow{P_0P_1}$ that corresponds to a_n. The length and the direction of the vector $\overrightarrow{P_kP_{k+1}}$ that corresponds to the coefficient a_{n-k} is obtained with the following equation:

$$(a_{n-k})e^{i(k)\pi/2}\ \text{, for } 0\leq k \leq n. \qquad (1)$$

The vector $\overrightarrow{P_kP_{k+1}}$ that corresponds to the coefficient a_{n-k} is a vector with the initial point P_k and the terminal point P_{k+1}. To make sense of the values given by (1), we must establish some rules. If the value of equation (1) is a positive real number, then the point P_{k+1} should be located to the right of P_k. If the value is a negative real number, then P_{k+1} is located directly to the left of P_k. If (1) yields a positive complex number, then P_{k+1} is directly above P_{n-k}. Finally, if (1) yields a negative complex number, then P_{k+1} is directly below P_k. In all the cases the length of the vector $\overrightarrow{P_kP_{k+1}}$ is equal to the absolute value of the coefficient a_{n-k}. For convenience we can call the value given by equation (1) the *vector value.*

To make matters more concrete, we can graph the polynomial $P(z)= 2z^2 + 2z -4$, that has $a_2 = 2$, $a_1 = 2$, $a_0 = -4$ and n = 2. First, we let P_0 to be located at (0,0). The direction of $\overrightarrow{P_0P_1}$ corresponding to the coefficient $a_{2-0} = a_2 = 2$ is given by the equation $(2)e^{i(0)\pi/2} = (2)e^0 = 2$. Since the vector value is a positive real number, we know that P_1 is 2 units to the right of P_0. Thus, the coordinates of P_1 are (2,0). The direction for $\overrightarrow{P_1P_2}$ that corresponds to $a_{2-1} = a_1 = 2$ is given by $(2)e^{i(1)\pi/2} = (2)e^{i\pi/2} = 2i$. Since we have a positive complex number, P_2 should be located 2 units above P_1 at the coordinates (2,2). Finally, for a_0 we have the vector $\overrightarrow{P_2P_3}$ with the direction

$(-4)e^{i(2)\pi/2} = (-4)\ e^{i\pi} = 4$. Thus, P_3 should be 4 units to the right of P_2, at the coordinates (6,2).

Figure 1 shows the Lill representation of the polynomial.

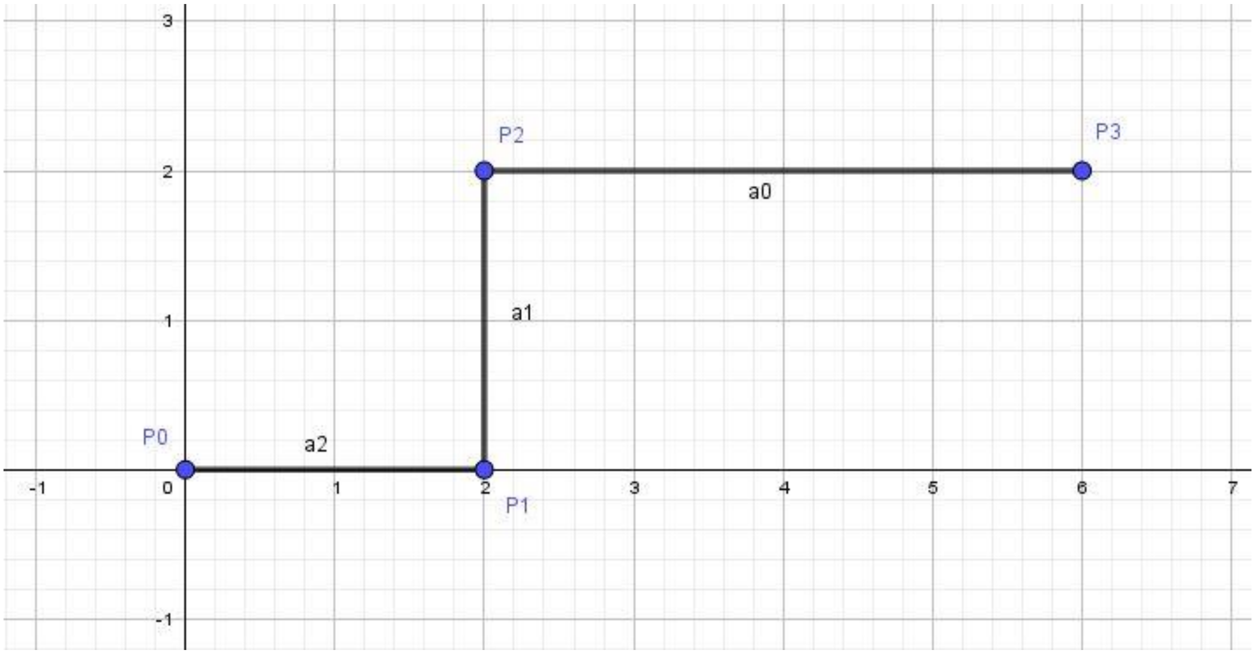

Figure 1

Equation (1) is not the only way to obtain the direction of vectors $\overrightarrow{P_k P_{k+1}}$ We can use a few rules of thumb. Starting with $\overrightarrow{P_1 P_2}$, the direction of $\overrightarrow{P_k P_{k+1}}$ is 90 degrees counterclockwise with respect to $\overrightarrow{P_{k-1} P_k}$ if a_{n-k} and a_{n-k-1} have the same sign. If a_{n-k} and a_{n-k-1} have different signs, then the direction of $\overrightarrow{P_k P_{k+1}}$ is 90 degrees clockwise with respect to $\overrightarrow{P_{k-1} P_k}$. In our example a_1 has the same sign as a_2, so the direction of $\overrightarrow{P_1 P_2}$ is 90 degrees counterclockwise with respect to $\overrightarrow{P_0 P_1}$. a_0 and a_1 have different signs, so the direction of $\overrightarrow{P_2 P_3}$, is 90 degrees clockwise with respect to $\overrightarrow{P_1 P_2}$,.

Graphing Polynomials with Complex Coefficients

Now that we understand how to graph polynomials with real coefficients, we can discuss the polynomials with complex coefficients. For polynomials with complex coefficients, the vector value of the vector $\overrightarrow{P_k P_{k+1}}$ that corresponds to the coefficient a_{n-k} is obtained with the following equation:

$$Re(a_{n-k})e^{i(k)\pi/2} + Im(a_{n-k})e^{i(k+1)\pi/2}, \text{ where } 0 \leq k \leq n \text{ and i is the imaginary number (2)}$$

It is easy to see that equation (2) is similar to equation (1). The big difference is that the equation (2) has two components. The first component gives the direction of the vector component that corresponds to the real component of a_{n-k} and the second component gives the direction of the vector component that corresponds to the imaginary part of a_{n-k}. $Re(a_{n-k})$ is the real part of the complex coefficient a_{n-k} and $Im(a_{n-k})$ is the imaginary part, such that $a_{n-k} = Re(a_{n-k}) + i\, Im(a_{n-k})$. For convenience, we can call the first component of the vector value the real component, and the second component we call the imaginary component. The directions given by the vector value obtained using equation (2), follow the same rules we established for equation (1).

To make matters more concrete, we can represent the 2nd degree polynomial $P(z)=z^2+z+(1-i)$. Thus we have $a_2=1$, $a_1=1$ and $a_0=1-i$. The vector $\overrightarrow{P_0 P_1}$ corresponding to $a_2=1$ is given by $Re(1)e^{i(0)\pi/2} + Im(1)e^{i(0+1)\pi/2}=e^0+0=1$. If we let P_0 to be at (0,0), then P_1 is one unit to the right of P_0 at (1,0). Now P_1 is the initial point for the vector $\overrightarrow{P_1 P_2}$ that corresponds to $a_1=1$. Similarly, the direction of vector $\overrightarrow{P_1 P_2}$ is given by $Re(1)e^{i(1)\pi/2} + Im(1)e^{i(1+1)\pi/2}= e^{i\pi/2}+0=i$.Thus P_2 is one unit up with respect to P_1, at the coordinates (1,1). Finally, we obtain the vector $\overrightarrow{P_2 P_3}$that corresponds to the coefficient $a_0=1-i$ and is given by $Re(1-i)e^{i(2)\pi/2} + Im(1-i)e^{i(2+1)\pi/2}=e^{i\pi} + (-1)e^{3i\pi/2}= -1 + (-1)(-i)= -1 + i$. Thus, P_3 is one unit to the left and one unit up with respect to P_2, at the coordinates (0,2). The Lill representation of the polynomial P(z) is shown in Figure 2.

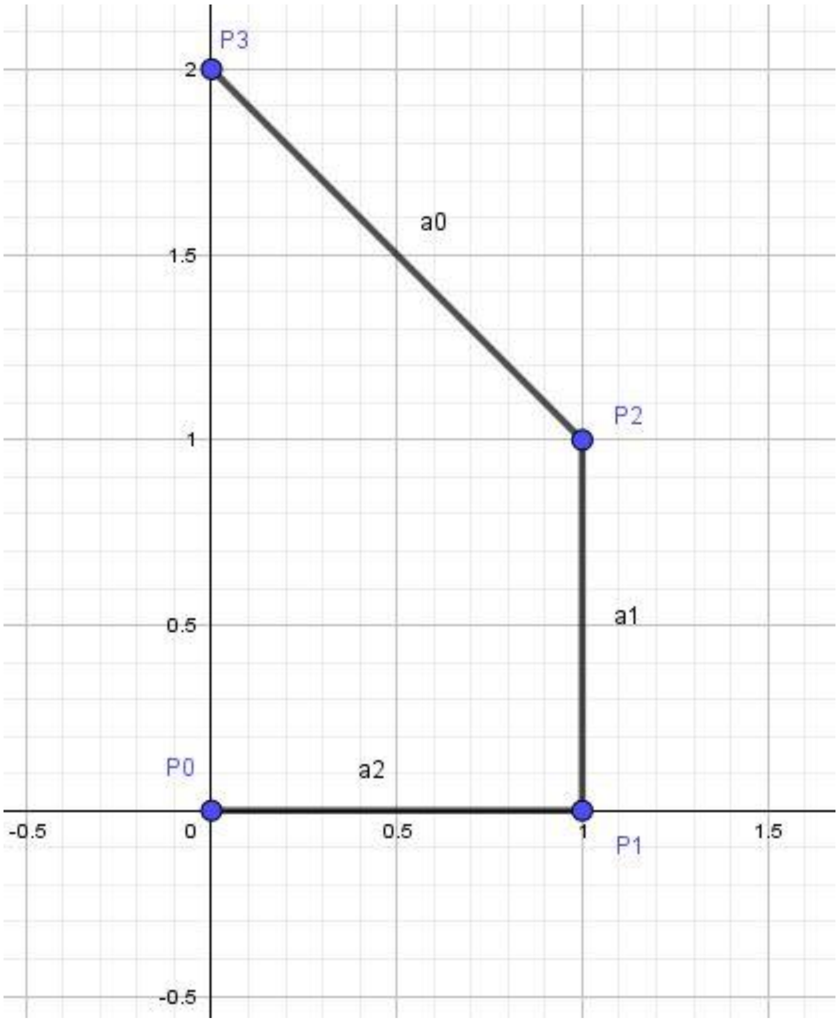

Figure 2

In the previous section we discussed the fact that two consecutive vectors $\overrightarrow{P_{k-1}P_k}$ and $\overrightarrow{P_kP_{k+1}}$ are usually at a 90-degree angle rotation with respect to each other when they represent two coefficients that are real. In this case a_2 and a_1 are real coefficients, so $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_1P_2}$ are at a 90-degree angle rotation with respect to each other. However, a_0 is a complex number with a real and an imaginary component, so the angle rotation between the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$ is no longer 90 degrees.

We can also devise a few rules of thumb for the complex coefficients. If the real part of a_{n-k} has the same sign as the imaginary part of a_{n-k}, then the component of $\overrightarrow{P_kP_{k+1}}$ that represents the imaginary part of a_{n-k} is 90 degrees counterclockwise with respect to the component that represents the real part. If the signs are different, then the angle is 90 degrees clockwise. In the case of $a_0 = 1 - i$, the signs are different. Using equation (2), we saw that the real part has a value of -1 and the imaginary part has a value of i. The component of $\overrightarrow{P_2P_3}$ that represents the real part of a_0 points towards the left, while the component that represents the imaginary part of a_0 points up (the location of P_3 was one unit to the left and one unit up with respect to P_2). Thus, the component of $\overrightarrow{P_2P_3}$ that represents the imaginary part is 90 degrees clockwise with respect to the component that represents the real part.

The rules of thumb presented in this section can be combined with the rules presented in the previous section to estimate the direction of each vector $\overrightarrow{P_kP_{k+1}}$ without the need of equations (1) and (2).

Graphing Solutions

In the general case when z is not necessarily a root of the polynomial P(z), the Lill path of variable z is given by $P_0A_1A_2\ldots A_{n-1}A_n$, such that the triangles $P_0P_1A_1$, $A_1P_2A_2$, $A_2P_3A_3$, …, $A_{n-2}P_{n-1}A_{n-1}$ and $A_{n-1}P_nA_n$ are similar and the angles $m(P_1P_0A_1)=m(P_2A_1A_2)=\ldots=m(P_nA_{n-1}A_n)=\theta$, where θ is the *associated angle* of z and $-180 \leq \theta \leq 180$. The associated angle θ is positive if $\overrightarrow{P_0A_1}$ is counterclockwise with respect to $\overrightarrow{P_0P_1}$. The variable z is a root only when A_n is the same point as P_{n+1}. We can adapt some formulas from paper [3], to obtain the exact location of the points A_1, A_2,… and A_n. Thus, taking the vector $\overrightarrow{P_kP_{k+1}}$ corresponding to a_{n-k} as our reference, we get:

$$\overrightarrow{A_kP_k} \text{ represents } H_{k,k}(z) = a_nz^k + a_{n-1}z^{k-1} + \ldots + a_{n-k+1}z \text{ with respect to } \overrightarrow{P_kP_{k+1}} \quad (3)$$

and

$$\overrightarrow{A_kP_{k+1}} \text{ represents } H_{k,k+1}(z) = a_nz^k + a_{n-1}z^{k-1} + \ldots + a_{n-k+1}z + a_{n-k} \text{ with respect to } \overrightarrow{P_kP_{k+1}} \quad (4)$$

So, we obtain these useful equations:

$\overrightarrow{A_1P_1}$ represents $H_{1,1}(z) = a_n z$, with respect to $\overrightarrow{P_1P_2}$ (5)

and

$\overrightarrow{A_nP_{n+1}}$ represents $H_{n,n+1}(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = P(z)$, with respect to $\overrightarrow{P_nP_{n+1}}$ (6)

$\overrightarrow{A_kP_k}$ and $\overrightarrow{A_kP_{k+1}}$ are vectors that have the initial point A_k. Using equation (3), the length of vector $\overrightarrow{A_kP_k}$ should be given by the absolute value of the polynomial $H_{k,k}(z)$. When we define the direction of $\overrightarrow{A_kP_k}$ with respect to $\overrightarrow{P_kP_{k+1}}$, we mean that both vectors have the same frame of reference for their real and imaginary components. For example, using equation (2) we know that the real component of $\overrightarrow{P_1P_2}$ is always parallel to the y-axis and that the positive direction is the direction that points up. The imaginary component of $\overrightarrow{P_1P_2}$ should be parallel to the x-axis and the positive direction is the direction that points to the left. From equations (3)-(4), we know that $\overrightarrow{A_1P_1}$ and $\overrightarrow{A_1P_2}$ should have the same frame of reference as $\overrightarrow{P_1P_2}$. Since $\overrightarrow{A_kP_k}$ and $\overrightarrow{P_kP_{k+1}}$ have the same frame of reference, we can modify equation (2) to obtain the following equation for the vector value of $\overrightarrow{A_kP_k}$:

$$\mathrm{Re}(H_{k,k}(z))e^{i(k)\pi/2} + \mathrm{Im}(H_{k,k}(z))e^{i(k+1)\pi/2},\text{ where } 1 \leq k \leq n \text{ and i is the imaginary number} \quad (7)$$

Formula (7) also gives the direction for $\overrightarrow{A_kP_{k+1}}$ if we replace $H_{k,k}(z)$ with $H_{k,k+1}(z)$. Since the location of P_k or P_{k+1} is already known, it is usually easier to determine the location of A_k using the vector values of vectors $\overrightarrow{P_kA_k}$ or $\overrightarrow{P_{k+1}A_k}$, which are the opposite of $\overrightarrow{A_kP_k}$ and $\overrightarrow{A_kP_{k+1}}$. The vector value of $\overrightarrow{P_kA_k}$ is obtained by multiplying the vector value of $\overrightarrow{A_kP_k}$ by -1.

To make the matters more concrete, we can graph the solution path of our polynomial $P(z)=z^2+z+(1-i)$, that was represented in the Figure 2. P(z) can be factored as $(z-i)(z+(1+i))$, so $z_1=i$ and $z_2=-1-i$. For convenience, Figure 3 shows the path for both roots. The path for $z_1=i$ is given by $P_0A_1P_3$ (the path has blue segments in Figure 3), where A_1 has the coordinates (2,0). $\overrightarrow{A_1P_1}$ represents $H_{1,1}$ (i) = a_2 i = i with respect to $\overrightarrow{P_1P_2}$. Using formula (7), the direction of $\overrightarrow{A_1P_1}$ is given by $\mathrm{Re}(i)e^{i(1)\pi/2} + \mathrm{Im}(i)e^{i(1+1)\pi/2} = 0 + (1)\, e^{i\pi} = -1$. Thus, P_1 should be one unit to the left of A_1. Formula (7) gave us the right direction, since A_1 is at (2,0) and P_1 is at (1,0). As mentioned in the previous paragraph, it is also useful to think about the vector value of vector $\overrightarrow{P_1A_1}$, which has the opposite direction of $\overrightarrow{A_1P_1}$. The vector value of $\overrightarrow{P_1A_1}$ is equal to the vector value of $\overrightarrow{A_1P_1}$ multiplied by -1. In our example, the vector value of $\overrightarrow{P_1A_1}$ is (-1)(-1)=1, so the vector points one unit to the right. So, we know that the point A_1 is one unit to the right of P_1. This approach is useful when we don't have the path of variable z plotted as we have in Figure 3. Once we have the Lill representation of the polynomial, as in Figure 2, we can use this approach to plot each point A_k. As we mentioned in the previous paragraph, this method is convenient since we already know the location of each point P_k, so it should be easy to determine the location of A_k once we know the vector value of $\overrightarrow{P_kA_k}$.

$\overrightarrow{A_1P_2}$ represents $H_{1,2}$ (i) = a_2 i + a_1 = i +1 with respect to P_1P_2. The vector value of $\overrightarrow{A_1P_2}$ is given $\mathrm{Re}(i+1)e^{i(1)\pi/2} + \mathrm{Im}(i+1)e^{i(1+1)\pi/2} = e^{i\pi/2} + e^{i\pi} = i - 1$. Thus, the vector points one unit up and one unit to the left. This is correct since A_1 is at (2,0) and P_2 is at (1,1). Since z_1 is a root, we know that $A_2 = P_3$. $\overrightarrow{A_2P_2}$ represents $H_{2,2}$ (i) = $a_2 (i)^2 + a_1(i) = -1 + i$. To determine the vector value of $\overrightarrow{A_2P_2}$ we can use equation (7), or we can use the fact that $\overrightarrow{A_2P_2} = \overrightarrow{P_3P_2}$. In the previous section we determined that the vector value of $\overrightarrow{P_2P_3}$ is -1 + i. Since $\overrightarrow{P_3P_2}$ goes in the opposite direction, its vector value is 1 -i. This is correct, since P_3 is at (0,2) and P_2 is at (1,1). Finally, $\overrightarrow{A_2P_3} = \overrightarrow{P_3P_3}$ and it represents P(i)=0.

The path for $z_2 = -1 - i$ is given by the green path $P_0B_1P_3$, where B_1 has the coordinates (0,1). We let the reader check if the path matches with the equations (3)-(7).

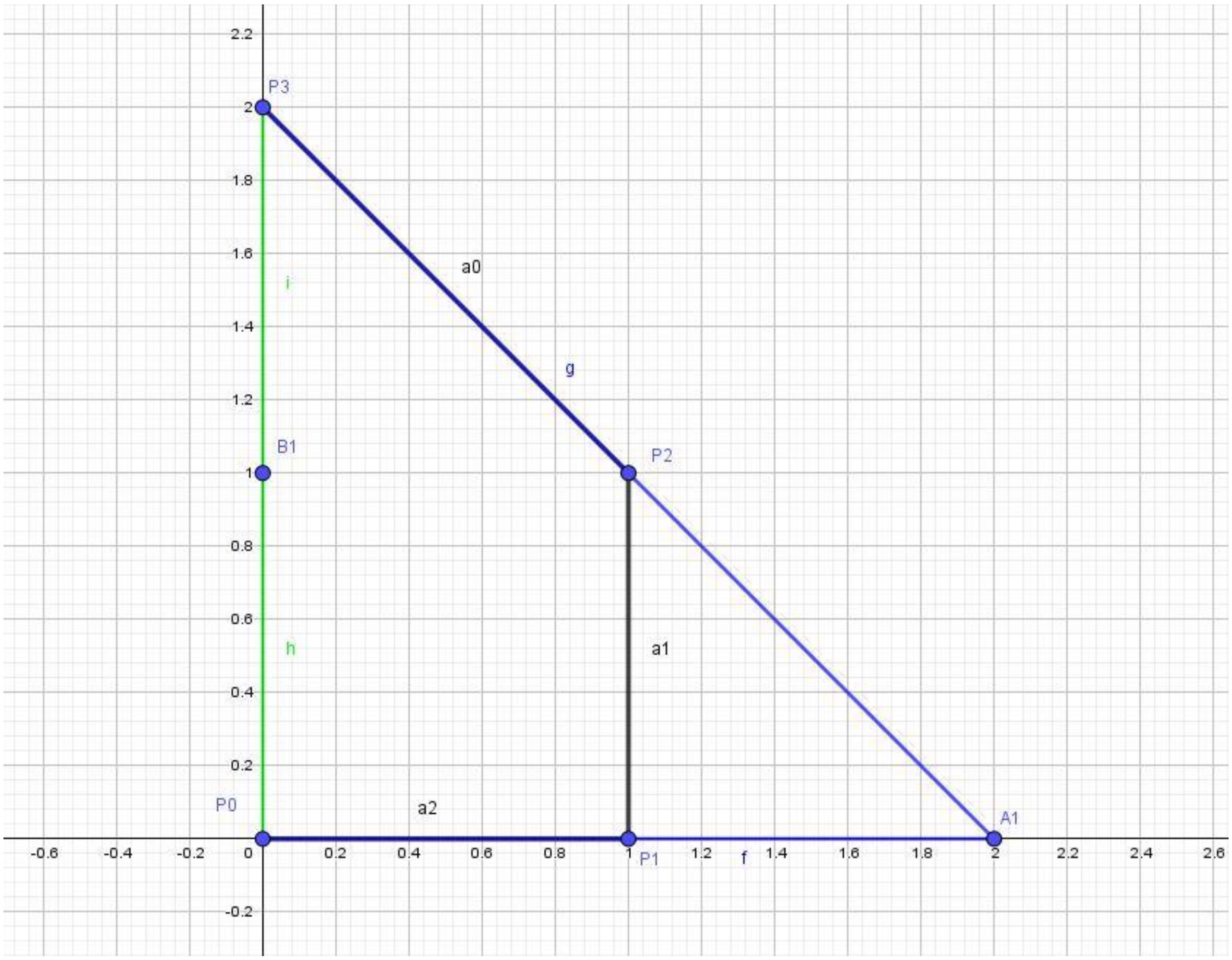

Figure 3

To check the correctness of our paths, we can also use the geometrical properties mentioned at the beginning of this section. We mentioned that each z has an associated angle θ. For z_1 we have the associated angle $\theta_1 = m(P_1P_0A_1) = m(P_2A_1P_3) = 0$. We also mentioned that the triangles $P_0P_1A_1$ and $A_1P_2P_3$ should be similar. Since $\theta_1 = 0$, the two triangles are degenerate triangles with collinear vertex points. Nonetheless, it can be checked that the sides of the two degenerate triangles are proportional and that indeed the two triangles are similar. The associated angle of z_2 is $\theta_2 = m(P_1P_0B_1) = m(P_2B_1P_3) = 90$ (P_0B_1 is 90 degrees counterclockwise with respect to P_0P_1), so we get the right triangles $P_0P_1B_1$ and $B_1P_2P_3$. The case of z_2 is much easier since the two triangles are identical.

Before we finish this section, we should make a few comments about the polynomials $H_{k,k}$ (z) and $H_{k,k+1}$ (z). The two polynomials can be compared to Horner's method for evaluating polynomials. In fact, we can say that the Lill path of z is a graphic representation of Horner's method. For convenience we can refer to the polynomials $H_{k,k}$ (z) and $H_{k,k+1}$ (z) as *Horner's companion polynomials*.

General Properties

In paper [4] it is shown that the path for a root z_r given by $P_0A_1A_2\ldots A_{n-1}P_{n+1}$ describes a polynomial of order *n-1* that has the same roots as $\frac{P(z)}{(z-z_r)}$ .The polynomial $P(z)/(z-z_r)$ is a polynomial of degree *n-1* that will have the same roots as P(z), not counting z_r. For example, in Figure 3 the path $P_0A_1P_3$ describes the polynomial $P_1(z)= 2z + (2+2i)$. The only root of $P_1(z)$ is $z_r=-1-i$, which is equal to z_2. The polynomial $P_1(z)$ has the same root as $P(z)/(z-z_1)$ The path $P_0B_1P_3$ describes the polynomial $P_2(z)= iz + 1$. The only root of $P_2(z)$ is $z_r=i$, which is equal to z_1. Again, $P_2(z)$ has the same root as $P(z)/(z-z_2)$. Thus, due to this *deflationary property* the degree of the polynomial P(z) can be reduced each time we find a new root. From our examples, we can see that this property also works for polynomials with complex coefficients.

In paper [5] it is shown that polynomials divisible by z^2+1 have a *closed Lill path*. A closed Lill path is a Lill path where P_0 coincides with P_{n+1}. For example, the polynomials $z^2 + 1$ and $z^3 + z^2 + z +1$ have a closed Lill path. In the most general case, it can be observed that a polynomial P(z) has a closed Lill path if and only if P(z) is divisible by (z+i). z^2+1 can be factored as (z + i)(z - i) and the paper [5] only considers polynomials with real coefficients. Due to the complex conjugate root theorem, if a polynomial P(z) with real coefficients has -i as a root it will also have i as a root. However, a polynomial with complex coefficients doesn't necessarily have pairs of complex conjugate roots. The simplest case of a polynomial with closed Lill path is P(z) = z + i, that has the root z_r = -i. Thus, in the most general case, the Lill path of a polynomial P(z) with complex coefficients is closed only if z = -i is a root of P(z). This anomaly is derived from the fact that the path of z = -i is always stationary. Thus, for z = -i, $P_0 = A_1 = A_2 = \ldots = A_n$ and if z= -i is a root, then $A_n = P_{n+1}$. Also, the associated angle of z = -i is undefined since $m(P_1P_0A_1)$ cannot be defined. Due to the anomaly at z = -i, a polynomial with a closed Lill path cannot be reduced to a lower degree polynomial using the stationary path of z = -i. Thus, the deflationary property doesn't work for z_r = -i.

Another observable general property for Lill graphs involves the associated angles of the roots of a polynomial. We can consider the general polynomial of order n $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, with the roots $z_1,z_2,...,z_n$. Each of these n roots has an associated angle $\theta_1,\theta_2,\ldots,\theta_n$. It can be observed that $m(P_1P_0P_{n+1})= \theta_1 + \theta_2 +\ldots \theta_{n..}$ To our knowledge, this *sum of angles property* is not discussed in other papers that deal with Lill's method. Thus, there is no proof for this property. We can see that the property works for the polynomial discussed in the previous section since $m(P_1P_0P_3) = \theta_1 + \theta_2 = 0 + 90 = 90$. From paper [2], we know that when the polynomial has only real coefficients and z is a real number, then z = - tan (θ). Using the trigonometric equation from [2] and Vieta's formulas, it is not hard to prove that $\tan(P_1P_0P_{n+1}) = \tan(\theta_1 + \theta_2 +\ldots \theta_n)$ when P(z) is a polynomial with real coefficients and *n* real roots. When all roots are real we can obtain an equation for $\tan(\theta_1 + \theta_2 +\ldots \theta_n)$ that is similar to the sum of angles formula presented in paper [6]. The situation becomes more difficult when we are dealing with complex roots and especially when we are dealing with polynomials with complex roots.

Also, it should be obvious that the sum of angles property doesn't work if the polynomial has a closed Lill path. In a closed Lill path $P_0 = P_{n+1}$, so $m(P_1P_0P_{n+1})$ is undefined. Nonetheless, the sum of angles property is an interesting property that in a way it seems to complement Vieta's formulas. While Vieta's formulas are algebraic, the sum of angles property is geometric or trigonometric in nature. Since we will not attempt to prove the sums of angles property in this paper, we will leave it as a conjecture.

There are other general properties related to the Lill graphs of polynomial equations. In paper [4] it is shown that Lill's method can be used to calculate the derivative of a polynomial at a specific z. In fact, the paper shows that for a nth order polynomial, you can calculate the nth order derivative for a specific z. The paper also discusses a few additional properties, however we prefer to let the readers explore these properties on their own.

Final Notes

In this paper we showed that Lill's method can be extended to represent polynomial equations with complex coefficients. We also discussed a few general properties that are very important when dealing with Lill's method and we also introduced new terminology. By extending the Lill's method, we also gained a better understanding of the mechanics behind this method.

We hope that the general properties discussed in this paper show that Lill's method enhances our understanding of the geometry of polynomials. We already mentioned that the *sum of angles property* is an interesting geometric or trigonometric property that complements Vieta's formulas. The *deflationary property* seems to be a graphic representation of the *polynomial remainder theorem*. We already mentioned that the Lill path for z is a geometric representation of Horner's method. In fact, many aspects of Lill's method seem to be the geometric equivalent of various algebraic or arithmetic methods, techniques or theorems. There are probably many other undiscovered properties that may prove to be useful for our understanding of polynomial equations.

The usefulness of Lill's method goes beyond the properties discussed in this paper. For example, there are certain techniques that can be used to solve quadratic equations. In paper [7], the author mentions a method to solve quadratics with real roots and a method to solve quadratics with complex roots. Maybe in the future somebody will discover similar techniques to deal with quadratics with complex roots. Paper [8] shows the relationship between Lill's method and origami techniques to solve cubic equations.

In the future, Lill's method may prove to be a useful educational tool. The beauty of Lill's method is that it shows that geometry, calculus, trigonometry, algebra and arithmetic are connected mathematical fields. Lill's method can be used to educate students to see mathematics as a more unified body of knowledge. The techniques and formulas presented in [2], [7] and [8] should be a good starting point for finding a way to implement Lill's method in an educational setting. To make Lill's method easier to understand, some formulas like equation (1) should be replaced by the rules of thumb presented in this paper. There are other changes that can be done to make Lill's method a more useful educational tool. Lill's method is a flexible method. Lill's method should also work well with geometric software like GeoGebra, so it can make the teaching experience more interactive.

Lill's method was discovered more than 150 years ago, but it is still an obscure or relatively obscure method. We hope that this paper showed that Lill's method deserves to be known by a larger audience. This method can be used to see many geometric properties of polynomial equations and we also believe that it has the potential to be used as an effective educational tool.

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